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An infinite transfer matrix approach to the product of random 2 × 2 positive matrices

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Abstract
This paper concerns the efficient and precise determination of the Laypunov exponent (and other statistical properties) of a product of random 2×2 matrices. By considering the ensemble average of an infinite series of regular functions and its iteration, we construct a transfer matrix, which is shown to be a trace class operator in a Hilbert space given that the positiveness of the random matrices is assumed. This fact gives a theoretical explanation of the superior convergence of the cycle expansion of the Lyapunov exponent (Bai 2007 *J. Phys. A: Math. Theor.* **40** 8315). A numerical method based on the infinite transfer matrix is applied to a one-dimensional Ising model with a random field and a generalized Fibonacci sequence. It is found that, in the presence of continuous distribution of a disorder or degenerated random matrix, the transfer matrix approach is more efficient than the cycle expansion method.

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1. Introduction

Problems in a one-dimensional disorder system often reduce to determining the asymptotic properties of a product of random matrices (PRM):

$$M_n = A_n \cdots A_2 A_1, \qquad \text{as} \quad n \to \infty, \tag{1}$$

where A_k's are random matrices from some specified distribution. For example, in a one-dimensional random Ising model [1],

$$A_k = \begin{bmatrix} e^{\beta(J+h_k)} & e^{\beta(-J+h_k)} \\ e^{\beta(-J-h_k)} & e^{\beta(J-h_k)} \end{bmatrix}, \tag{2}$$

where h_k is a random variable characterizing the external field, and β and J are two constants which are related to the free energy by

$$F(\beta, J) = -\lim_{n\to\infty} \frac{1}{n\beta} \langle \log \mathrm{Tr}(M_n) \rangle, \tag{3}$$

where $\langle\, , \rangle$ denotes the ensemble average. Other well-known examples can be found in the randomly coupled harmonic oscillators proposed by Dyson [2] or the quantum mechanics of an electron in a one-dimensional disorder potential [3], where the density of state or the length of localization can be derived from a properly defined PRM. In addition, PRM has also been widely used to model discrete stochastic processes such as the evolution of population [4] and investment strategy [5].

The most fundamental theorem on PRM was given by Furstenberg and Kesten [6], which states that a definite exponential decay or growth can be expected for almost every realization of M_n, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \log\|M_n\| \to \gamma \tag{4}$$

with unit probability. $\|.\|$ denotes any matrix norm. The characteristic number γ is called the Lyapunov exponent and can be often endowed with an important physical meaning. For example, in the random Ising model, obviously we have $F(\beta, J) = -\gamma/\beta$. Despite the well-established theory and few exact results under a certain elaborate choice of the ensemble of random matrices (e.g. [7–9]), for a general PRM the accurate calculation of γ presents a considerable numerical challenge in practice. (For details, see [10, 11] and references therein.) In 1992, Mainieri derived a cycle expansion of the Lyapunov exponent and demonstrated that it converges exponentially with the cycle length [12, 13]. This novel method originates from the periodic orbit theory in the study of chaotic dynamics [14, 15], and, with a recent improvement, it even enjoys a super-exponential convergence when the positiveness of the random matrices is assumed [16].

Here we study an alternative method for evaluateing the Lyapunov exponent, namely the transfer matrix approach. The motivation is of twofold: on one hand, we aim at a theoretical understanding of the superior convergence of cycle expansion and on the other hand, we desire to design a practical algorithm for the statistical analysis of a PRM. The idea of the transfer matrix is in fact quit natural. Consider a sequence of random vectors generated by a PRM,

$$\{\mathbf{x}_0, M_1\mathbf{x}_0, M_2\mathbf{x}_0, M_3\mathbf{x}_0, M_4\mathbf{x}_0 \ldots\} \equiv \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \ldots\}. \tag{5}$$

When $n \to \infty$, it is well known that the direction of $\mathbf{x}_n$, i.e. $\theta_n = \mathbf{x}_n/|\mathbf{x}_n|$, will establish an equilibrium distribution $\rho(\theta)$, based on which one can derive γ (and other statistical properties) of this PRM [11]. For numerical approximation of a dynamically generated equilibrium distribution, a method suggested by Ulam is widely used due to its conceptual simplicity [17]. The key idea of the Ulam method is as follows. By dividing θ-space into N small parts, the random recurrence $\theta_n \to \theta_{n+1}$ can be represented by a $N \times N$ transfer matrix, of which the eigenvector with the leading eigenvalue can be used as a good approximation of $\rho(\theta)$. For example, this method was once used by Embree and Trefethen in their study of a generalized Fibonacci sequence [18]. However, numerically Ulam's method could be quite inefficient, which is in particular the case when an ensemble of positive random matrices is assumed. In this case, $\theta_n \to \theta_{n+1}$ is given by the random iteration of some contraction mappings and hence the invariant distribution is a highly irregular function, just as the fractal generated by an iterated function system [19]. The irregularity of $\rho(\theta)$ implies that the ensemble average is well defined only for regular functions (or physical observables) of θ. Therefore, in order to achieve a better numerical efficiency, it is more reasonable to represent the random evolution of θ by the iteration of the ensemble average of a set of regular functions, i.e. a transfer matrix in the dual space of the probability density functions.

In this paper we shall show that, when only the positive 2×2 random matrices are concerned, the above-described transfer matrix can readily be constructed. In terms of the obtained transfer matrix, we can rewrite the cycle expansion of γ, and attribute its excellent convergence to the fact that this transfer matrix is a trace class operator in a Hilbert space. Moreover, a numerical method based on this transfer matrix can be, under a certain circumstance, more convenient and efficient than that based on the conventional cycle expansion method. The remainder of this paper is organized as follows. In the following section, we explicitly construct the transfer matrix associated with the product of one matrix and show that it is a trace class operator. In section 3, the transfer matrix is applied to a PRM and the related computational method is described in detail. The numerical efficiency of the transfer matrix method is examined in section 4, which is followed by a brief discussion in section 5.

2. Transfer matrix

In this section, we first define the transfer matrix associated with a positive 2×2 matrix. After that, based on a qualitative analysis of a linear fractional transformation, this infinite matrix is shown to be in the trace class and its trace and spectrum are determined accordingly.

Consider a 2×2 matrix $A = \{a_{ij}\}$ and a two-dimensional vector $\mathbf{x} = (x_1, x_2)^T$, $a_{ij}, x_1, x_2 > 0$. Define

$$|\mathbf{x}| = x_1 + x_2 \qquad \text{and} \qquad z = \frac{x_1 - x_2}{x_1 + x_2}. \tag{6}$$

$\mathbf{x}' = A\mathbf{x}$ induces

$$\frac{|\mathbf{x}'|}{|\mathbf{x}|} = \hat{a}_{21}z + \hat{a}_{22} \equiv h_A(z) \qquad \text{and} \qquad z' = \frac{\hat{a}_{11}z + \hat{a}_{12}}{\hat{a}_{21}z + \hat{a}_{22}} \equiv f_A(z), \tag{7}$$

where $\hat{A} \equiv \{\hat{a}_{ij}\}$ is a similarity transformation of A that is defined as

$$\hat{A} = \frac{1}{2}\begin{bmatrix} a_{11} - a_{12} - a_{21} + a_{22}, & a_{11} + a_{12} - a_{21} - a_{22} \\ a_{11} - a_{12} + a_{21} - a_{22}, & a_{11} + a_{12} + a_{21} + a_{22} \end{bmatrix}. \tag{8}$$

Obviously, we have

$$f_A \circ f_B = f_{AB} \qquad \text{and} \qquad (h_A \circ f_B)h_B = h_{AB}. \tag{9}$$

The dimensionless variable z can be regarded as a coordinate in θ-space mentioned in our previous discussion. Define an infinite vector

$$\phi^q(\mathbf{x}) = |\mathbf{x}|^q(1, z, z^2, \ldots)^T, \tag{10}$$

and write the iteration of ϕ^q under the action of A as

$$\phi^q(A\mathbf{x}) = \mathcal{L}_q(A)\phi^q(\mathbf{x}). \tag{11}$$

In this definition, q is a free parameter whose meaning will be clear in the following section. Simple calculation shows that the elements of $\mathcal{L}_q(A)$ are given by the expansion

$$h_A^q(z)f_A^k(z) = \frac{(\hat{a}_{11}z + \hat{a}_{12})^k}{(\hat{a}_{21}z + \hat{a}_{22})^{k-q}} = \sum_{j=0}^{\infty}\mathcal{L}_q(A)_{kj}z^j \tag{12}$$

$k = 0, 1, \ldots$. We call $\mathcal{L}_q(A)$ the *transfer matrix* induced from A.

According to the definition of $\mathcal{L}_q(A)$, formally we have

$$\mathcal{L}_q(B)\mathcal{L}_q(A) = \mathcal{L}_q(BA). \tag{13}$$

However, this relation holds only when the following infinite sum

$$\sum_{s=0}^{\infty}\mathcal{L}_q(B)_{ks}\mathcal{L}_q(A)_{sj} \tag{14}$$

converges for all $k, j \geqslant 0$. An important property of $\mathcal{L}_q(A)$, which we shall show later, is that its elements are exponentially bounded, namely

$$|\mathcal{L}_q(A)_{kj}| \leqslant c\eta^{k+j} \tag{15}$$

$(0 < \eta < 1)$. Hence equation (13) is a rigorously identity; in other words, $A \to \mathcal{L}_q(A)$ is a semigroup homomorphism. In fact, the exponential decay of matrix elements implies that $\mathcal{L}_q(A)$ is a trace class operator in a Hilbert space, or $\mathcal{L}_q(A) \in \mathcal{J}_1$. With many excellent properties, trace class operators behave much like finite dimensional matrices. For example, $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{J}_1$ implies $\mathcal{A}_1\mathcal{A}_2 \in \mathcal{J}_1$ and $c_1\mathcal{A}_1 + c_2\mathcal{A}_2 \in \mathcal{J}_1$. Their traces are well defined, which are representation independent, and $\mathrm{tr}(\mathcal{A}_1\mathcal{A}_2) = \mathrm{tr}(\mathcal{A}_2\mathcal{A}_1)$. Moreover, if $\mathcal{A} \in \mathcal{J}_1$, then

$$\det(1 - z\mathcal{A}) = 1 - \mathrm{tr}(\mathcal{A})z - \tfrac{1}{2}(\mathrm{tr}(\mathcal{A}^2) - (\mathrm{tr}(\mathcal{A}))^2)z^2 + \cdots \tag{16}$$

is an entire analytical function of z (see e.g., [20] for more details).

Now let us prove the inequality given by equation (15). Note that $f_A(z)$ is the fractional linear transformation associated with $\hat{A}$, which maps a circle to a circle in the complex plane. Especially, the image of a disk $D_r = \{z, |z| \leqslant r\}$ under the action of f_A is a disk centered at $(f_A(r) + f_A(-r))/2$ with radius $|f_A(r) - f_A(-r)|/2$ if $r < |z_p|$, where $z_p = -\hat{a}_{22}/\hat{a}_{21}$ is the pole of $f_A(z)$. Since $|z_p| > 1$, we have $f_A(D_1) \subseteq D_s, s = \max\{|f_A(1)|, |f_A(-1)|\} < 1$. From the viewpoint of continuity, there exists $0 < \eta < 1$ that satisfies $f_A(D_{\frac{1}{\eta}}) \subseteq D_\eta$. Next, to represent the matrix elements as a closed loop integration,

$$\mathcal{L}_q(A)_{kj} = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{h_A^q(z) f_A^k(z)}{z^j} \frac{\mathrm{d}z}{z}, \tag{17}$$

and denote the boundary of $D_{\frac{1}{\eta}}$ as C, we have

$$|\mathcal{L}_q(A)_{kj}| = \frac{1}{2\pi} \left| \oint_{|\eta z|=1} \frac{h_A^q(z) f_A^k(z)}{z^j} \frac{\mathrm{d}z}{z} \right| \leqslant c\eta^{k+j}, \tag{18}$$

where $c = \max\left\{ \left|h_A^q(z)\right| : |\eta z| = 1 \right\} < \infty$.

By using the integral representation of the matrix elements, we can readily calculate the trace of $\mathcal{L}_q(A)$:

$$\mathrm{tr}(\mathcal{L}_q(A)) = \sum_{k=0}^{\infty} \mathcal{L}_q(A)_{kk} = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{h_A^q(z)}{z - f_A(z)}\, \mathrm{d}z = \frac{\lambda_0^q}{1 - \lambda_1/\lambda_0}, \tag{19}$$

where λ_0 (λ_1) is the larger(smaller) eigenvalue of A. From

$$\begin{aligned} \det(1 - z\mathcal{L}_q(A)) &= \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\left(\mathcal{L}_q^n(A)\right) \right] = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_q(A^n)) \right] \\ &= \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{\lambda_0^{qn}}{1 - (\lambda_1/\lambda_0)^n} \right] \\ &= \exp\left[-\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\left(z\lambda_0^{q-k}\lambda_1^k\right)^n}{n} \right] = \Pi_{k=0}^{\infty}\left(1 - z\lambda_0^{q-k}\lambda_1^k\right), \end{aligned} \tag{20}$$

we find that the spectrum of $\mathcal{L}_q(A)$ consists of a geometric series

$$\lambda_0^q, \qquad \lambda_0^{q-1}\lambda_1, \qquad \lambda_0^{q-2}\lambda_1^2, \qquad \lambda_0^{q-3}\lambda_1^3, \ldots. \tag{21}$$

Moreover, when $q = n$ is a non-negative integer, as

$$h_A^n(z) f_A^k(z) = (\hat{a}_{11}z + \hat{a}_{12})^k (\hat{a}_{21}z + \hat{a}_{22})^{n-k} \tag{22}$$

is an n-order polynomial of z if $0 \leqslant k \leqslant n$, $\mathcal{L}_n(A)$ takes the form

$$\mathcal{L}_n(A) = \begin{bmatrix} \mathcal{L}_n^{(0)}(A) & | & \mathbf{0} \\ --- & | & --- \\ * & | & \mathcal{L}_n^{(1)}(A) \end{bmatrix}. \tag{23}$$

The upper left $(n + 1) \times (n + 1)$ block, i.e. $\mathcal{L}_n^{(0)}(A)$, can be identified with the restriction of $\bigotimes^n A$, i.e. the direct product of n replica of A, in the symmetric subspace, and it contributes to the spectrum of $\mathcal{L}_n(A)$ the first $n + 1$ eigenvalues.

3. Product of random matrices

In this section, we apply the transfer matrix to a PRM. We first review the definition of the generalized Lyapunov exponents and relate it to the leading eigenvalue of the transfer matrix. Then we discuss some details of the numerical method based on the transfer matrix.

Suppose the random matrices are independently sampled from $\{T_1, T_2, \ldots, T_m\}$ with preassigned probabilities: $\mathrm{prob}(T_i) = p_i$, $\sum_{i=1}^m p_i = 1$. As in the last section, we assume T_i's are positive 2×2 matrices. To characterize the divergent behavior of the PRM, one can introduce a generalized Lyapunov exponent $\tau(q)$, which is defined according to

$$\langle |\mathbf{x}_n|^q \rangle \sim \exp[n\tau(q)], \tag{24}$$

when $n \to \infty$, or

$$\tau(q) = \lim_{n\to\infty} \frac{1}{n} \log(\langle |\mathbf{x}_n|^q \rangle), \tag{25}$$

where $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \ldots$ is a stochastic vector sequence generated from this PRM (see equation (5)). As the value of $\tau(q)$ is independent of the specific choice of vector norm, for the simplicity we fix $|\mathbf{x}| = x_1 + x_2$ for positive vectors. The $\tau(q)$ curve is more informative than the Lyapunov exponent γ. For example, γ can be derived from $\tau(q)$ according to

$$\gamma = \left.\frac{\partial \tau(q)}{\partial q}\right|_{q=0}, \tag{26}$$

and, moreover,

$$\left.\frac{\partial^2 \tau(q)}{\partial q^2}\right|_{q=0} = \lim_{n\to\infty} \frac{1}{n}[\langle (\log|\mathbf{x}_n|)^2 \rangle - (\langle \log|\mathbf{x}_n| \rangle)^2] \equiv \chi, \tag{27}$$

which characterizes the typical fluctuation of $\log|\mathbf{x}_n|$.

Noting that

$$\begin{aligned}
\langle \phi^q(\mathbf{x}_1) \rangle &= \sum_{i=1}^m p_i \mathcal{L}_q(T_i) \phi^q(\mathbf{x}_0) \\
\langle \phi^q(\mathbf{x}_2) \rangle &= \sum_{i,j=1}^m p_i p_j \mathcal{L}_q(T_i T_j) \phi^q(\mathbf{x}_0) = \sum_{i,j=1}^m p_i p_j \mathcal{L}_q(T_i) \mathcal{L}_q(T_j) \phi^q(\mathbf{x}_0) \\
&= \left[\sum_{i=1}^m p_i \mathcal{L}_q(T_i)\right]^2 \phi^q(\mathbf{x}_0) \qquad (28) \\
&\ldots \\
\langle \phi^q(\mathbf{x}_n) \rangle &= \left[\sum_{i=1}^m p_i \mathcal{L}_q(T_i)\right]^n \phi^q(\mathbf{x}_0),
\end{aligned}$$

we conclude that $\phi^q(\mathbf{x}_n) \sim \lambda_q^n \psi^*$ when $n \to \infty$ and hence $\tau(q) = \log \lambda_q$, where λ_q and ψ^* are the leading eigenvalue and its corresponding eigenvector of the transfer matrix

$$\mathcal{L}_q = \langle \mathcal{L}_q(T) \rangle = \sum_{i=1}^{m} p_i \mathcal{L}_q(T_i). \tag{29}$$

As the traces of $\mathcal{L}_q$ and its powers can exactly be calculated, e.g.

$$\mathrm{tr}(\mathcal{L}_q) = \sum_i \frac{p_i \lambda_0^q(T_i)}{1 - \lambda_1(T_i)/\lambda_0(T_i)}, \tag{30}$$

$$\mathrm{tr}\left(\mathcal{L}_q^2\right) = \sum_i \frac{\left[p_i \lambda_0^q(T_i)\right]^2}{1 - [\lambda_1(T_i)/\lambda_0(T_i)]^2} + 2 \sum_{i<j} \frac{p_i p_j \lambda_0^q(T_i T_j)}{1 - \lambda_1(T_i T_j)/\lambda_0(T_i T_j)} \tag{31}$$

and so on, according to equation (16) we can expand $\det(1 - z\mathcal{L}_q)$ at $z = 0$ to any given order, i.e.

$$\det(1 - z\mathcal{L}_q) = \sum_{k=0}^{n} a_k z^k + o(z^n) \equiv P_n(z; q) + o(z^n). \tag{32}$$

Then, from the smallest zero of $P_n(z; q)$ we can obtain an estimation of λ_q. Along this line, a cycle expansion of γ is thus derived (see [16] for detail). Because $\mathcal{L}_q$ is a trace class operator, and hence $\det(1 - z\mathcal{L}_q)$ is an entire function of z, the error induced by replacing $\det(1 - z\mathcal{L}_q)$ with $P_n(z; q)$ is super-exponentially small when $n \to \infty$. This explains the superior convergence of the cycle expansion method.

In the following, we consider a more direct numerical approach: approximating the transfer matrix $\mathcal{L}_q$ by its first $(N_c + 1) \times (N_c + 1)$ block. With the exponential decay of the matrix elements of $\mathcal{L}_q$, it is reasonable to expect a high numerical efficiency for this straightforward method. We begin with the evaluation of the matrix elements. Consider a general expansion

$$g(z)\left(\frac{c_{11}z + c_{12}}{c_{21}z + c_{22}}\right)^k = \sum_{i=0}^{\infty} w(k, j) z^j. \tag{33}$$

Obviously, $g(z) = \sum_j w(0, j) z^j$ and $w(k, 0) = g(0)(c_{12}/c_{22})^j$. Noting that

$$(c_{21}z + c_{22}) g(z) \left(\frac{c_{11}z + c_{12}}{c_{21}z + c_{22}}\right)^{k+1} = (c_{11}z + c_{12}) g(z) \left(\frac{c_{11}z + c_{12}}{c_{21}z + c_{22}}\right)^k, \tag{34}$$

the remaining elements can be calculated by the recursion:

$$w(k + 1, j + 1) = \alpha w(k, j) + \beta w(k, j + 1) + \delta w(k + 1, j), \tag{35}$$

where $(\alpha, \beta, \delta) = (c_{11}, c_{12}, -c_{21})/c_{22}$.

The leading eigenvalue λ_q can be obtained by the standard power method, which also yields the corresponding (right and left) eigenvectors,

$$\mathcal{L}_q |q\rangle = \lambda_q |q\rangle \qquad \text{and} \qquad \langle q| \mathcal{L}_q = \lambda_q \langle q| \quad (\langle q|q\rangle = 1). \tag{36}$$

So the derivative of λ_q can be calculated according to

$$\frac{\partial \lambda_q}{\partial q} = \langle q| \frac{\partial \mathcal{L}_q}{\partial q} |q\rangle. \tag{37}$$

For instance, if $q = 0$, then $\lambda_0 = 1$ and $\langle 0| = [1, 0, 0, \ldots]$. Assuming

$$|0\rangle = [1, v_1, v_2, \ldots]^T, \tag{38}$$

Table 1. Convergence of the truncated transfer matrix method for the statistics of a PRM related to a random Ising chain. N_c denotes the cut-off dimension.

N_c	γ	χ
0	0.933	0.0
2	0.928 19	0.009 5
4	0.928 127	0.009 753
6	0.928 125 87	0.009 758
8	0.928 125 843	0.009 759 10
10	0.928 125 842 94	0.009 759 110 0
12	0.928 125 842 915	0.009 759 110 244
14	0.928 125 842 914 3	0.009 759 110 249 9
16	0.928 125 842 914 35	0.009 759 110 250 15

we have

$$\gamma = \langle 0|\frac{\partial \mathcal{L}_0}{\partial q}|0\rangle = \langle \log \hat{t}_{22}\rangle + \sum_{j=1}^{\infty}(-1)^{j+1}\left\langle\left(\frac{\hat{t}_{21}}{\hat{t}_{22}}\right)^j\right\rangle\frac{v_j}{j}. \tag{39}$$

To calculate χ, we need the second-order derivative of λ_q, which is a little bit lengthy:

$$\frac{\partial^2\lambda_q}{\partial q^2} = \langle q|\frac{\partial^2\mathcal{L}_q}{\partial q^2}|q\rangle + \frac{2}{\lambda_q}\langle q|\frac{\partial \mathcal{L}_q}{\partial q}\mathcal{D}\frac{\partial \mathcal{L}_q}{\partial q}|q\rangle, \tag{40}$$

where

$$\mathcal{D} = \sum_{t=0}^{\infty}\mathcal{P}\left(\frac{\mathcal{L}_q}{\lambda_q}\right)^t \tag{41}$$

and $\mathcal{P} = I - |q\rangle\langle q|$ is a projection operator. For an arbitrary right vector $\phi, \mathcal{D}\phi$ can be achieved from the fixed point of the iteration

$$\psi \to \mathcal{P}\left(\frac{\mathcal{L}_q}{\lambda_q}\psi + \phi\right). \tag{42}$$

4. Applications

Our first example is the random Ising model (see equations (1)–(3)). For the sake of simplicity, we fix $\beta = 2J = 1$ and consider two ensembles of the random field. The first one is $h = \pm 1/2$ with equal probabilities. In this case, the cycle expansion works very well. When $\det(1 - z\mathcal{L}_q)$ is expanded to z^{11}, it yields more than 20 converging digits, i.e. $\gamma = 0.928\,125\,842\,914\,359\,104\,554\ldots$. As a comparison, the results of the truncated transfer matrix are summarized in table 1 as well, from which we can see that very precise estimation of the statistics of a PRM can be achieved with a rather small truncated matrix[1]. As another comparison, we also perform a Monte Carlo simulation of this disorder system. Based on K realizations of a product of L random matrices, γ and its uncertainty can be estimated. The result (table 2) indicates that the precision of the Monte Carlo simulation is approximately given by $(\chi/LM)^{1/2}$, which is much poorer than the transfer matrix (and cycle expansion [16]) method.

[1] For the two examples we discussed in this section, by taking advantage of the symmetry of $z \to -z$, the dimensionality of truncated transfer matrices can be reduced to $1 + N_c/2$.

Table 2. The Lyapunov exponent obtained from Monte Carlo simulation; the number in parenthesis denotes the standard uncertainty in the last two digits.

	$M = 10^2$	$M = 10^3$	$M = 10^4$
$L = 10^3$	0.928 14(31)	0.928 18(10)	0.928 144(29)
$L = 10^4$	0.928 12(10)	0.928 089(32)	0.928 115(10)
$L = 10^5$	0.928 112(31)	0.928 133(10)	0.928 127 4(31)
$L = 10^6$	0.928 130(10)	0.928 127 0(31)	0.928 127 6(10)

Table 3. Discrete approximation of a PRM with continuous randomness. $\gamma_{k,0}$ is the Lyapunov exponent of a product of 2^k random matrices, calculated by the transfer matrix method. $\gamma_{k,1} = \frac{1}{3}(4\gamma_{k,0} - \gamma_{k-1,0})$ and $\gamma_{k,2} = \frac{1}{15}(16\gamma_{k,1} - \gamma_{k-1,1})$.

k	$\gamma_{k,0}$	$\gamma_{k,1}$	$\gamma_{k,2}$
1	0.928 125 842 914 35		
2	0.951 574 940 612 31	0.959 391 306 511 63	
3	0.957 288 115 946 76	0.959 192 507 724 90	0.959 179 254 472 45
4	0.958 707 458 515 03	0.959 180 572 704 46	0.959 179 777 036 43
5	0.959 061 742 327 94	0.959 179 836 932 24	0.959 179 787 880 76
6	0.959 150 278 912 20	0.959 179 791 106 95	0.959 179 788 051 94
7	0.959 172 410 912 09	0.959 179 788 245 38	0.959 179 788 054 61
8	0.959 177 943 777 95	0.959 179 788 066 58	0.959 179 788 054 66
9	0.959 179 326 986 04	0.959 179 788 055 40	0.959 179 788 054 66

Next, we consider a uniform probability distribution of $h \in [-1, 1]$. In this continuous case, evaluation of $\mathrm{tr}\left(\mathcal{L}_q^n\right)$ turns out to be an n-fold integration (see equations (30)–(31)), which causes a severe limitation in computing the cycle expansion to high order. The transfer matrix method, in contrast, works almost equally well with the infinite choices of the random matrix. To simulate the continuous distribution of randomness, we assume

$$h = h_i = \frac{2\mathrm{i} - 1 - m}{m}, \qquad i = 1, 2, \ldots, m \tag{43}$$

with equal probabilities. When m runs from 2 to about 10^3, 14 converging digits of γ can always be achieved with $N_c = 30$. Therefore, the increase of m results in only a linear increase of the computational cost in the evaluation of matrix elements. The numerical results at $m = 2^k$ are summarized in table 3; with an acceleration of convergence by Romberg's method, we obtain a very accurate estimation of the Lyapunov exponent. While for a Monte Carlo simulation with $M = 10^3$ and $L = 10^5$, it gives $\gamma = 0.959\,176 \pm 1.7 \times 10^{-5}$.

Our second example is a generalized random Fibonacci sequence proposed by Embree and Trefethen [18]: $x_0 = x_1 = 1$ and $x_{n+1} = x_n \pm \alpha^2 x_{n-1}$ for $n \geqslant 1$, where the sign before α^2 is equiprobably refreshed at each step. Although $\langle x_n \rangle = x_1$ remains fixed, for a typical realization of this random sequence we have $x_n \sim \mathrm{e}^{n\gamma}$ as $n \to \infty$. It was found that, when $\alpha > 1/2$, γ shows an intriguing fractal-like dependence upon α [18]. On the other hand, when $\alpha < 1/2$, γ is an analytical function [21]:

$$\gamma = -\frac{1}{2}\alpha^4 - \frac{7}{4}\alpha^8 - \frac{29}{3}\alpha^{12} - \frac{555}{8}\alpha^{16} - \frac{2843}{5}\alpha^{20} - \frac{30755}{6}\alpha^{24} - \cdots. \tag{44}$$

In the original derivation of this weak disorder expansion, it was hard to obtain the high order coefficients due to the exponential increase of computational cost. Here we first adopt the idea of the transfer matrix to give a more efficient derivation of this expansion.

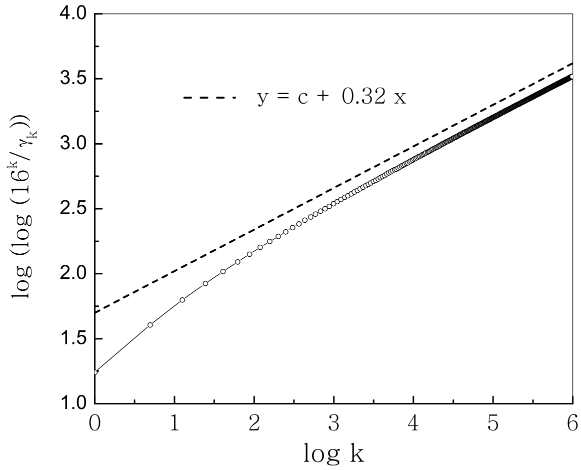

Figure 1. Coefficients of the weak disorder expansion of the Lyapunov exoponent for the random sequence $x_{n+1} = x_n \pm \alpha^2 x_{n-1}$. The first 400 terms are plotted (hollow circles), which suggests that $\gamma_k \sim 16^k \exp(-ck^{0.32})$.

Letting

$$u_k = [x_k - (1-\alpha)x_{k-1}, (1+\alpha)x_{k-1} - x_k]^T, \tag{45}$$

this model can be rewritten as $u_{k+1} = T_1 u_k$ or $T_2 u_k$, where

$$T_1 = \begin{bmatrix} \frac{1}{2}+\alpha & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-\alpha \end{bmatrix} \qquad \text{and} \qquad T_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}-\alpha \\ \frac{1}{2}+\alpha & \frac{1}{2} \end{bmatrix}. \tag{46}$$

If $\phi_0 = [1, v_1, v_2, \ldots]^T$ is the right eigenvector of $\mathcal{L}_0$, $\mathcal{L}_0\phi_0 = \phi_0$ implies $v_{2k+1} = 0$ and

$$v_{2k} = \alpha^{2k}\left[1 + \sum_{j=1}^{\infty} \frac{(2k+2j-1)!}{(2k-1)!(2j)!}\alpha^{2j} v_{2j}\right] \tag{47}$$

for $k > 0$. Rewriting it as

$$v_{2k} = \alpha^{2k}\sum_{j=0}^{\infty} c(k, j)\alpha^{4j}, \tag{48}$$

we have $c(0, k) = \delta_{k0}$, $c(k, 0) = 1$ and

$$c(k, j) = \sum_{s=1}^{j} \frac{(2k+2s-1)!}{(2k-1)!(2s)!} c(s, j-s) \tag{49}$$

for $k, j > 0$. All $c(k, j)$'s can be recursively obtained from the above relation and, according to equation (39), we have

$$\gamma = -\sum_{k=1}^{\infty} \frac{\alpha^{2k} v_{2k}}{2k} = -\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} \frac{c(j, k-j)}{2j}\right)\alpha^{4k} \equiv -\sum_{k=1}^{\infty} \gamma_k \alpha^{4k}. \tag{50}$$

With the help of Mathematica, several hundred terms of γ_k can thus exactly be calculated. The divergence of γ_k as $k \to \infty$ is plotted in figure 1, which suggests that

$$\gamma_k \sim 16^k \exp(-c_1 k^{c_2}) \qquad \text{when} \quad k \to \infty, \tag{51}$$

Table 4. Decay rates ($\sigma = e^{\gamma}$) and typical fluctuation exponents (χ) of the random sequence $x_{n+1} = x_n \pm \alpha^2 x_{n-1}$ for various α.

α^2	σ	χ
1/128	0.999 969 476 366 27	0.000 061 056 275 52
1/64	0.999 877 832 701 48	0.000 244 478 980 26
1/32	0.999 510 160 763 98	0.000 982 005 014 65
1/16	0.998 021 538 991 47	0.003 995 235 468 23
1/8	0.991 752 822 024 32	0.017 188 718 823 47
1/4	0.957 974 454 585 06	0.109 697 994 520 99

where $c_1 \approx 4.968$ and $c_2 \approx 0.32$. If this asymptotic behavior is qualitatively correct, then for any positive integer n, $\frac{d^n\gamma}{d\alpha^n}$ is well defined when $\alpha \to 1/2$ from below.

As in the previous example, here the transfer matrix method gives a rapidly convergent estimation of the statistics of the generalized Fibonacci sequence when $\alpha < 1/2$. All the results listed in table 4, except for $\alpha = 1/2$, converge at $N_c < 20$ and all the digits of σ agree with those obtained by the weak disorder expansion. As a comparison, we note that a θ-space transfer matrix of very large dimensionality (2^{20}) produces only six correct digits [18].

Finally, we consider the case of $\alpha = 1/2$, where T_2 is a degenerated matrix in more detail. As $\mathrm{tr}(\mathcal{L}_q(T_2))$ and hence $\mathrm{tr}(\mathcal{L}_q)$ are no longer well defined, the cycle expansion must take its origin form [12]. For the transfer matrix method, although its convergence is relatively slow (14 stable digits of γ can be obtained when $N_c = 100$), it is still much efficient than the cycle expansion method and the Monte Carlo simulation. Furthermore, we calculate the whole spectrum of the finite approximation of $\mathcal{L}_0$ and focus on how they behave when $\alpha \to 1/2$. The eigenvalues are organized in a descending order, namely

$$1 = \lambda^{(0)} \geqslant \lambda^{(1)} \geqslant \lambda^{(2)} \geqslant \lambda^{(3)} \cdots, \tag{52}$$

and a few of the leading eigenvalues are plotted in figure 2, from which we can see that they approach a fixed point at 1/2 in a manner guided by

$$\frac{1}{2}\left[\frac{1-\sqrt{1-4\alpha^2}}{1+\sqrt{1-4\alpha^2}}\right]^k, \tag{53}$$

$k = 1, 2, \ldots,$ i.e. the spectrum of $\frac{1}{2}\mathcal{L}_0(T_1)$. Such aggregation of eigenvalues is a typical manifestation of intermittency in dynamical systems [22], which slows down the mixing rate and results in a power law decay of correlation. Fortunately, the situation in our case is less severe: there is a finite gap between the leading eigenvalue of $\mathcal{L}_0$ and that of its degenerate component, i.e. $\frac{1}{2}\mathcal{L}_0(T_1)$. Therefore the leading eigenvector is well-defined and the exponential convergence of the power method is retained in this marginal case.

5. Summary and discussion

In this paper, we have studied how to obtain the statistics of a product of random positive 2×2 matrices by employing the transfer matrix method. Unlike Ulam's method which concerns how an equilibrium distribution (of vector direction) is established under the action of random matrices, the transfer matrix suggested here describes how the steady ensemble average of a set of physical observables is reached. The transfer matrix is shown to be a trace class operator in a Hilbert space, and this fact gives a theoretical understanding of the rapid convergence of the cycle expansion method [16]. In addition, the property of trace class implies that this

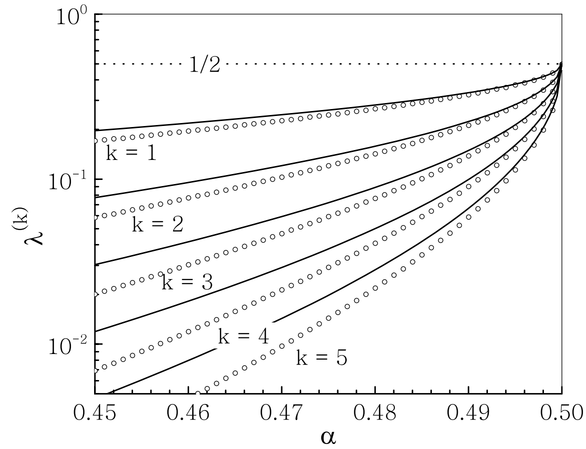

Figure 2. Spectrum of the transfer matrix ($\mathcal{L}_0$) for the random sequence $x_{n+1} = x_n \pm \alpha^2 x_{n-1}$. The eigenvalues are calculated with $N_c = 1000$ and the five next-to-leading ones, i.e. $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(5)}$, are plotted (hollow circles). For comparison, also plotted the corresponding eigenvalues of $\frac{1}{2}\mathcal{L}_0(T_1)$(solid lines).

infinite transfer matrix can be well approximated by a finite one. As demonstrated by two numerical examples, to truncate the infinite transfer matrix into one of quite low dimension is sufficient to generate a surprisingly high accurate estimation of the statistics of a PRM.

One of the most important discoveries in the study of chaotic dynamics is that global quantities such as the Lyapunov exponent, the rate of correlation decay and the quantum levels can efficiently be extracted from the whole spectrum of periodic orbits in a systematic manner, i.e. cycle expansion or periodic orbit theory [15]. The validity of cycle expansion relies on the trace class property of a certain transfer matrix. Therefore, a rigorous proof of this property is of great importance for understanding of the global dynamics of nonlinear systems (e.g. [23, 24]). As a typical stochastic dynamical system, it is interesting to ask whether the transfer matrix theory developed here can be extended to PRM in high dimensional cases? The answer is not immediate. For example, if A is an arbitrary 3×3 positive matrix, by analogy with the 2×2 case, we can define a transfer matrix $\mathcal{L}_q(A)$ according to

$$\frac{(\hat{a}_{11}z + \hat{a}_{12}w + \hat{a}_{13})^k(\hat{a}_{21}z + \hat{a}_{22}w + \hat{a}_{23})^l}{(\hat{a}_{31}z + \hat{a}_{32}w + \hat{a}_{33})^{k+l-q}} = \sum_{j,m=0}^{\infty} \mathcal{L}_q(A)_{kl;jm} z^j w^m, \tag{54}$$

where $\hat{A} = \{\hat{a}_{ij}\} = TAT^{-1}$ with T being a fixed coordinates transformation. But, however, it is hard to generalize the argument in section 2 to show that the above-defined infinite matrix is a trace class operator. It is not clear whether this difficulty is merely a technique one that can be circumvented by a more sophisticated mathematical method or an implication that a more restrictive condition is required to guarantee the trace class property of the transfer matrix in high dimensional cases. This question is worthy of further studies.

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